THE COMBINATORIAL PART OF THE COHOMOLOGY OF A SINGULAR VARIETY

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Abstract. We study the first step of the weight filtration on the cohomology of a proper complex algebraic variety, which we call the combinatorial part. We obtain a natural upper bound on its size, which gives rather strong information about the topology of rational singularities.

Given a possibly reducible complex algebraic variety X, we define the *combinatorial part* of the compactly supported cohomology to a subspace $KH^i_c(X) \subseteq H^i_c(X, \mathbb{Z})$ characterized by the following axioms:

(K1) These subspaces are preserved by proper pullbacks.

(K2) If X is smooth and complete, $KH^0_c(X) = H^0_c(X)$ and $KH^i_c(X) = 0$ for $i > 0$.

(K3) If $U \subseteq X$ is an open immersion and $Z = X - U$, then the standard exact sequence

$$\ldots H^{i-1}_c(Z) \to H^i_c(U) \to H^i_c(X) \to \ldots$$

restricts to an exact sequence

$$\ldots KH^{i-1}_c(Z) \to KH^i_c(U) \to KH^i_c(X) \to \ldots$$

The proof of uniqueness, when X is complete, given below is a simple induction. Existence will follow by identifying $KH^i_c(X)$ with the first step of the weight filtration $W_0H^i_c(X)$ of Deligne [D] and Gillet-Soulé [GS]. It will be both convenient and necessary to review the basic construction which gives a method for calculating this in terms of the underlying combinatorics of a simplicial resolution. In simple cases, such as when X has simple normal crossing singularities, this can be made quite explicit. We note that in this paper varieties are reduced schemes of finite type over $\mathbb{C}$. We can extend this an arbitrary complex scheme of finite type X, by defining $KH^i_c(X) = KH^i_c(X_{red})$.

Work of Stepanov [S] and the second author [B] suggested a certain natural bound on the dimension of the combinatorial part of cohomology of the exceptional divisor of a singularity. The main purpose of this note is to verify this in a refined form. Given a proper map of varieties $f : X \to Y$, we show that $\dim KH^i(f^{-1}(y))$ is bounded above by $\dim (R^if_*\mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y$. In particular, in accordance with a conjecture of Stepanov, the first space vanishes for a resolution of a rational singularity.

1. Uniqueness for complete varieties

As a warm up, we prove the uniqueness statement for complete varieties. For this it is convenient to replace (K3) by (K3′) below.

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Lemma 1.1. *Assume that KH_c^i satisfies the axioms (K1)-(K3). Given a complete variety X with closed set S and a desingularization $f : \tilde{X} \to X$ which is an isomorphism over $X - S$. Let $E = f^{-1}(S)$.*

(K3′) *Then there is an exact sequence*

$$\ldots \to KH^{i-1}(E) \to KH^i(X) \to KH^i(\tilde{X}) \oplus KH^i(S) \to \ldots$$

Proof. This follows from diagram chase on

$$\begin{tikzcd}
\ldots KH^{i-1}(S) \arrow[r] \arrow[d] & KH_c^i(U) \arrow[r] \arrow[d, "="] & KH^i(X) \arrow[r] \arrow[d] & KH^i(S) \ldots \arrow[d] \\
KH^{i-1}(E) \arrow[r] & KH_c^i(U) \arrow[r] & KH^i(\tilde{X}) \arrow[r] & KH^i(E)
\end{tikzcd}$$

□

Remark 1.2. *In general, by the same argument we get a sequence*

$$\ldots \to KH_c^{i-1}(E) \to KH_c^i(X) \to KH_c^i(\tilde{X}) \oplus KH_c^i(S) \to \ldots$$

Lemma 1.3. *Assume that KH^i satisfies the axioms (K1), (K2) and (K3′). Given a complete variety X with a closed set S and a desingularization $f : \tilde{X} \to X$ which is an isomorphism over $X - S$. Let $\tilde{S} \to S$ be a desingularization of S and $F = \tilde{S} \times_X \tilde{X}$. Then there is an exact sequence*

$$\ldots \to KH^{i-1}(F) \to KH^i(X) \to KH^i(\tilde{X}) \oplus KH^i(\tilde{S}) \to \ldots$$

Proof. Consider the diagram

$$\begin{tikzcd}
\tilde{F} \arrow[rr] \arrow[dr, "f"] \arrow[dd] & & \tilde{X} \times \tilde{S} \arrow[dr, "p"] \arrow[dd] & \\
& F \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
\tilde{S} \arrow[rr, "\gamma"] \arrow[dr, "id_{\tilde{S}}"] & & X \times \tilde{S} \arrow[dr, "p"] & \\
& \tilde{S} \arrow[rr] & & X
\end{tikzcd}$$

where the maps labelled p are projections, γ is the graph of the composition $\tilde{S} \to S \to X$, and $\tilde{F} = \tilde{S} \times_{X \times \tilde{S}} (\tilde{X} \times \tilde{S})$. The lefthand square containing f and id_s is easily seen to be Cartesian. Therefore f gives an isomorphism $\tilde{F} \cong F$. Thus from the previous lemma, we obtain an exact sequence

$$\ldots \to KH^{i-1}(F) \to KH^i(X \times S) \to KH^i(\tilde{X} \times S) \oplus KH^i(\tilde{S}) \to \ldots$$

Choose base points $s_1, \ldots s_N$ in each connected component of S, Define $\sigma = \frac{1}{N} \sum_j (id \times s_j)* : H^i(X \times S) \to H^i(X)$. This gives a left inverse to p^*. Then

a diagram chase using

$$\begin{tikzcd}
KH^{i-1}(F) \arrow[r, "\delta"] \arrow[d, dash, "="] & KH^i(X\times S) \arrow[r] \arrow[d, shift left, "\sigma"] & KH^i(\tilde X\times S)\oplus KH^i(\tilde S) \arrow[r] \arrow[d, shift left] & KH^i(F) \arrow[d, dash, "="] \\
KH^{i-1}(F) \arrow[r, "\sigma\circ\delta"] & KH^i(X) \arrow[r] \arrow[u, shift left, "p^*"] & KH^i(\tilde X)\oplus KH^i(\tilde S) \arrow[r] \arrow[u, shift left] & KH^i(F)
\end{tikzcd}$$

shows that the bottom row is exact.

□

Theorem 1.4. *There is at most one collection of subspaces $KH^i(X)\subseteq H^i(X,\mathbb{Z})$, with X complete, satisfying axioms (K1), (K2) and (K3′).*

Proof. We prove this by induction on i. First we check that $KH^0(X) = H^0(X)$. We can assume that X is connected. If $p\in X$, then

$$H^0(X) = H^0(p) = KH^0(p) = KH^0(X)$$

by the axioms.

By lemma 1.3, there is an exact sequence

$$KH^{i-1}(F)\to KH^i(X)\to KH^i(\tilde X)\oplus KH^i(\tilde S) = 0$$

So $KH^i(X) = im[KH^{i-1}(F)\to H^i(X)]$. □

2. Simplicial resolutions

The general construction is based on simplicial resolutions. We start by recalling some standard material [D, GNPP, PS]. A simplicial object in a category is a diagram

$$\ldots X_2 \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} X_1 \begin{smallmatrix}\to\\ \to\end{smallmatrix} X_0$$

with n face maps $\delta_i : X_n\to X_{n-1}$ satisfying the standard relation $\delta_i\delta_j = \delta_{j-1}\delta_i$ for $i<j$; this would be more accurately called a "strict simplicial" or "semisimplicial" object since we do not insist on degeneracy maps going backwards. The basic example of a simplicial set, i.e. simplicial object in the category of sets, is given by taking X_n to be the set of n-simplices of a simplicial complex on an ordered set of vertices. Let Δ^n be the standard n-simplex with faces $\delta_i' : \Delta^{n-1}\to\Delta^n$. Given a simplicial set or more generally a simplicial topological space, we can glue the $X_n\times\Delta^n$ together by identifying $(\delta_i x, y)\sim(x,\delta_i' y)$. This leads to a topological space $|X_\bullet|$ called the geometric realization, which generalizes the usual construction of the topological space associated a simplicial complex.

Given a simplicial space, filtering $|X_\bullet|$ by skeleta $\bigcup_{n\le N} X_n\times\Delta^n/\sim$ yields the spectral sequence

$$E_1^{pq} = H^q(X_p, A)\Rightarrow H^{p+q}(|X_\bullet|, A) \qquad (1)$$

for any abelian group A. It is convenient to extend this. A simplicial sheaf on $X_\bullet$ is a collection of sheaves $\mathcal{F}_n$ on X_n with "coface" maps $\delta_i^*\mathcal{F}_{n-1}\to\mathcal{F}_n$ satisfying the face relations. For example, the constant sheaves $\mathbb{Z}_{X_\bullet}$ with identities for coface maps forms a simplicial sheaf. If $X_\bullet$ is a simplicial object in the category of complex manifolds, then $\Omega^i_{X_\bullet}$ with the obvious maps, forms a simplicial sheaf. We can define cohomology by setting

$$H^i(X_\bullet,\mathcal{F}_\bullet) = Ext^i(\mathbb{Z}_{X_\bullet},\mathcal{F}_\bullet)$$

This generalizes sheaf cohomology in the usual sense, and it can be extended to the case where $\mathcal{F}_\bullet^\bullet$ is a bounded below complex of simplicial sheaves by using a hyper Ext. When $\mathcal{F} = A$ is constant, this coincides with $H^i(|X_\bullet|, A)$. But in general the meaning is more elusive. There is a spectral sequence

$$E_1^{pq}(\mathcal{F}_\bullet^\bullet) = H^q(X_p, \mathcal{F}_p^\bullet) \Rightarrow H^{p+q}(X_\bullet, \mathcal{F}_\bullet^\bullet) \tag{2}$$

generalizing (1). Filtering $\mathcal{F}^\bullet$ by the "stupid filtration" $\mathcal{F}_\bullet^{\geq n}$ yields a different spectral sequence

$$'E_1^{pq} = H^q(X_\bullet, \mathcal{F}_\bullet^p) \Rightarrow H^{p+q}(X_\bullet, \mathcal{F}_\bullet^\bullet) \tag{3}$$

Theorem 2.1 (Deligne). *If $X_\bullet$ is a simplicial object in the category of compact Kähler manifolds and holomorphic maps. The spectral sequence* (1) *degenerates at E_2 when $A = \mathbb{Q}$.*

Remark 2.2. *The theorem follows from a more general result in* [D, 8.1.9]. *However the argument is very complicated. Fortunately, as pointed out in* [DGMS], *this special case follows easily from the $\partial\bar{\partial}$-lemma. Here we give a more complete argument.*

Proof. It is enough to prove this after tensoring with $\mathbb{C}$. We can realize the spectral sequence as coming from the double $(E^\bullet(X_\bullet), d, \pm\delta)$, where $(E^\bullet, d)$ is the C^∞ de Rham complex, and δ is the combinatorial differential. (We are mostly going to ignore sign issues since they are not relevant here.) In fact this is a triple complex, since each $E^\bullet(-)$ is the total complex of the double complex $(E^{\bullet\bullet}(-), \partial, \bar{\partial})$.

Given a class $[\alpha] \in H^i(X_j)$ lying in the kernel of δ, we have $\delta\alpha = d\beta$ for some $\beta \in E^{i-1}(X_{j+1})$ Then $d_2([\alpha])$ is represented by $\delta\beta \in E^{i-1}(X_{j+2})$. We will show this vanishes in cohomology. The ambiguity in the choice of β will turn out to be the key point.

By the Hodge decomposition, we can assume that α is pure of type (p, q). Therefore $\delta\alpha$ is also pure of this type. We can now apply the $\partial\bar{\partial}$-lemma [GH, p 149] to write $\alpha = \partial\bar{\partial}\gamma$ where $\gamma \in E^{p-1,q-1}(X_{j+1})$. This means we have two choices for β. Taking $\beta = \bar{\partial}\gamma$ shows that $d_2([\alpha])$ is represented by a form of pure type $(p-1, q)$. On the other hand, taking $\beta = -\partial\gamma$ shows that this class is of type $(p, q-1)$. Thus $d_2([\alpha]) \in H^{p-1,q} \cap H^{p,q-1} = 0$.

By what we just proved $\delta\alpha = d\beta, \delta\beta = d\eta$, and $\delta\eta$ represents $d_3([\alpha])$. It should be clear that one can kill this and higher differentials in the exact same way.

□

Corollary 2.3. *With the same assumptions as the theorem, the spectral sequence* (2) *degenerates at E_2 when $\mathcal{F} = \mathcal{O}_{X_\bullet}$.*

(This fixes an incorrect proof in [S, 2.4].)

Proof. By the Hodge theorem, the spectral sequence for $\mathcal{F} = \mathcal{O}_{X_\bullet}$ is a direct summand of the spectral sequence for $\mathcal{F} = \mathbb{C}$. □

Theorem 2.4 (Deligne). *Given any (possibly reducible) variety X, there exists a smooth simplicial variety $X_\bullet$, which we call a* simplicial resolution*, with proper morphisms $\pi_\bullet : X_\bullet \to X$ (commuting with face maps) inducing a homotopy equivalence between $|X_\bullet|$ and X. Given a morphism $f : X \to Y$ there exists simplicial resolutions $X_\bullet, Y_\bullet$ and a morphism $f_\bullet : X_\bullet \to Y_\bullet$ compatible with f.*

The theorem is a consequence of resolution of singularities. Proofs can be found in [D, GNPP, PS]. Note that the original construction of Deligne results in a necessarily infinite diagram, whereas the method of Guillen et. al yields a fairly economical resolution. Here are some examples.

Example 2.5. *Suppose that X is an analytic space, whose irreducible components X^i are compact Kähler, and suppose that their intersections $X^{ij\cdots} = X^i \cap X^j \ldots$ are all smooth. This includes the case of a divisor with simple normal crossings. Then an explicit simplicial resolution is given by taking X_i to be the disjoint union of $(i+1)$-fold intersection of components of X. The face map δ_k is given by inclusions*

$$X^{i_1\ldots i_n} \subset X^{i_i\ldots \hat{i}_k \ldots i_n}\ (i_1 < \ldots < i_k)$$

Remark 2.6. *The above construction makes perfect sense for general X, and it yields a (generally singular) simplicial variety $X_\bullet$ with $|X_\bullet|$ homotopic to X. $X_\bullet$ will always be dominated by a simplicial resolution.*

Example 2.7. *Following the method of* [GNPP] *we can construct a simplicial resolution of a variety X with isolated singularities as follows. Let $f : \tilde{X} \to X$ be a resolution of singularities such that the exceptional divisor $E = \cup E^i$ is a divisor with simple normal crossings. Write $E^{ij} = E^i \cap E^j$ and $E_n = \coprod E^{i_0\ldots i_n}$. Let $S_0 \subset X$ be set of singular points, $S_1 \subseteq S_0$ be the set of images of $\cup E^{ij}$ and so on. Then the simplicial resolution is given by*

$$\ldots E_1 \sqcup S_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} E_0 \sqcup S_1 \rightrightarrows \tilde{X} \sqcup S_0$$

where the face maps are given by inclusions $S_i \to S_{i-1}$ on the second component. On the first component δ_k is given by

$$\begin{cases} E^{i_1\ldots i_n} \subset E^{i_i\ldots \hat{i}_k \ldots i_n} & \textit{if } k \le n \\ f : E^{i_1\ldots i_n} \to S_{n-1} & \textit{if } k = n+1 \end{cases}$$

Given a simplicial resolution, the spectral sequence (1) will then converge to $H^*(X, A)$. More generally for any sheaf, there is an isomorphism

$$H^i(X, \mathcal{F}) \cong H^i(X_\bullet, \pi_\bullet^* \mathcal{F})$$

for any sheaf $\mathcal{F}$ on X. The last property goes by the name of *cohomological descent.*

Given a closed subvariety $\iota : Z \subset X$, there exists simplicial resolutions $Z_\bullet \to Z$, $X_\bullet \to X$ and a morphism $\iota_\bullet : Z_\bullet \to X_\bullet$ covering ι. Then there is a new smooth simplicial variety $cone(\iota_\bullet)$ ([D, §6.3], [GNPP, IV §1.7]) whose geometric realization is homotopy equivalent to X/Z. So that the spectral sequence (1) converges to $H^*_c(X - Z, A)$. Although simplicial resolutions are far from unique, the filtration on $H^*_c(X - Z, A)$ is the weight filtration W [GS], and this is canonically determined by $X - Z$ alone. When $A = \mathbb{Q}$, this part of the datum of the canonical mixed structure.

Let X be a proper variety with a possibly empty closed set Z. Let $U = X - Z$. Choose a simplicial resolution $C_\bullet = Cone(Z_\bullet \to X_\bullet)$ as above. By convention W is an increasing filtration indexed so that

$$W_q H_c^{p+q}(U)/W_{q-1} = E_\infty^{pq}\ (\cong E_2^{pq} \text{ over } \mathbb{Q})$$

In particular, $W_{-1} = 0$. The part of interest W_0, can be computed as follows. We can form a simplicial set by applying the connected component functor π_0 to $C_\bullet$.

This simplicial set $|\pi_0(C_\bullet)|$ is called the dual complex or nerve of the simplicial resolution. We have

$$W_0H^i_c(U,\mathbb{Q}) \cong H^i(\ldots \to H^0(C_p,\mathbb{Q}) \to H^0(C_{p+1},\mathbb{Q})\ldots) \cong H^i(|\pi_0(C_\bullet)|,\mathbb{Q})$$

For integer coefficients, $W_0H^i(U,\mathbb{Z}) = \pi^*H^i(|\pi_0(C_\bullet)|,\mathbb{Z})$ where $\pi : C_\bullet \to \pi_0(C_\bullet)$ is the constant map on components. So that this piece of the filtration is determined by the underlying combinatorial information encoded by the dual complex.

Theorem 2.8. *There is a collection of subspaces $KH^i_c(X) \subseteq H^i_c(X,\mathbb{Z})$ satisfying axioms (K1)-(K3) given in the introduction. Moreover, it is uniquely characterized by axioms.*

Proof. For existence, we note that $KH^i_c(X) = W_0H^i_c(X)$ satisfies these axioms by [GS, §3.1]. (For rational coefficients, this goes back to [D].)

So it remains to check uniqueness. We already checked this when X is complete. The nonsingular case follows from this. If X is nonsingular, we can choose a nonsingular compactification $\bar{X}$. Then from the axioms, we get $KH^i_c(X) = im[KH^{i-1}(\bar{X} - X)]$. Then the general case now follows from the main theorem of [GN] together with remark 1.2.

□

From the formula $K = W_0$, we can deduce further properties.

Corollary 2.9. $KH^i_c(X \times Y) = \bigoplus_{j+k=i} KH^j_c(X) \otimes KH^k_c(Y)$

Proof. [GS, thm 3]. □

Corollary 2.10. *Let $\pi : \tilde{X} \to X$ be a resolution of a complete variety such that the exceptional divisor E has normal crossings. Let $S = \pi(E) \subset X$. Then $\dim KH^i(X)$ is the $(i-1)$st Betti number b_{i-1} of the dual complex of E when $i > 2\dim(S)+1$. If S is nonsingular, then this holds for $i > 1$. When $i = 2\dim(S)+1$, $\dim KH^i(X) = b_{i-1}$ minus the number of irreducible components of S of maximum dimension.*

Proof. This follows from lemma 1.1, the identification of $KH^i(E) = W_0H^i(E)$ and the above remarks. □

When X is a divisor with simple normal crossings, $KH^i_c(X)$ is the cohomology of the dual complex. As remarked earlier 2.6, we can use a construction to a build a simplicial variety canonically attached to X, for any X. If we apply π_0 to this simplicial variety, we get a simplicial set Σ_X canonically attached to X, that we will call the nerve or dual complex. There is a canonical map $H^i(|\Sigma_X|,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ coming from the spectral sequence (1) associated to this simplicial variety. From the discussion in 2.5 and 2.6, we can see that:

Lemma 2.11. *If X is complete, the image $H^i(|\Sigma_X|,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ lies in $KH^i(X,\mathbb{Q})$. If X satisfies the assumptions of example 2.5, then these subspaces coincide.*

3. Bounds on the combinatorial part

Suppose that X is a complete variety. Then in addition to the weight filtration $H^i(X,\mathbb{C})$ carries a second filtration, called the Hodge filtration induced on the abutment $H^i(X,\Omega^\bullet_{X_\bullet}) \cong H^i(X,\mathbb{C})$ of the spectral sequence (3) for $\Omega^\bullet_{X_\bullet}$. By convention F is decreasing. We have $F^0 = H^i(X,\mathbb{C})$ and

$$F^0H^i(X,\mathbb{C})/F^1 \cong H^i(X_\bullet,\mathcal{O}_{X_\bullet})$$

W induces the same filtration on the right as the one coming from (2). In particular,

$$W_0 Gr_F^0 H^i(X,\mathbb{C}) = H^i(\ldots \to H^0(X_p,\mathcal{O}) \to H^0(X_{p+1},\mathcal{O})\ldots)$$

$$\cong H^i(|\pi_0(X_\bullet)|,\mathbb{C}) \cong W_0 H^i(X,\mathbb{C})$$

This means that Hodge filtrations becomes trivial on $W_0H^i(X)$. So that this is a vector space and nothing more.

Theorem 3.1.
(a) *If X is a complete variety, then there is an inclusion $KH^i(X,\mathbb{C}) \hookrightarrow H^i(X,\mathcal{O}_X)$.*
(b) *If $f : X \to Y$ a proper morphism of varieties, then there is an inclusion $KH^i(f^{-1}(y),\mathbb{C}) \hookrightarrow (R^i f_*\mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y$ for each $y \in Y$.*

Proof. The canonical map κ factors

$$\begin{tikzcd} H^i(X,\mathbb{C}) \arrow[rr, "\kappa"] \arrow[dr] & & H^i(X,\mathcal{O}_X) \\ & Gr_F^0 H^i(X,\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

Thus

$$W_0H^i(X,\mathbb{C}) \subseteq Gr_F^0 H^i(X,\mathbb{C}) = im[H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X)]$$

which implies (a).

For (b), let X_y be the reduced fibre over y, and $X_y^{(n)}$ the fibre with its nth infinitesimal structure. From (a), we have a natural inclusion $s : W_0H^i(X_y,\mathbb{C}) \hookrightarrow H^i(X_y,\mathcal{O}_{X_y})$. After choosing a simplicial resolution of the fibre $f_\bullet : \mathcal{X}_\bullet \to X_y$, s can be identified with the composition

$$E_2^{i0}(\mathbb{C}) \to E_2^{i0}(\mathcal{O}_{\mathcal{X}_\bullet}) \to H^i(X_y,\mathcal{O}_{X_y})$$

where the first map is induced by the natural map $\mathbb{C} \to \mathcal{O}$, and the last map is the edge homomorphism. Applying the same construction to the simplicial sheaf $f_\bullet^*\mathcal{O}_{X_y^{(n)}}$ yields a map s_n fitting into a commutative diagram

$$\begin{tikzcd} W_0H^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[dr, "s_n"] & & H^i(X_y,\mathcal{O}_{X_y}) \\ & H^i(X_y,\mathcal{O}_{X_y^{(n)}}) \arrow[ur] & \end{tikzcd}$$

Furthermore, these maps are compatible, thus they pass to map s_∞ to the limit. Together with the formal functions theorem [H, III 11.1], this yields a commutative diagram

$$\begin{tikzcd} W_0H^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[d, "s_\infty"] \arrow[drr, "s'"] & & H^i(X_y,\mathcal{O}_{X_y}) \\ \varprojlim H^i(X_y,\mathcal{O}_{X_y^{(n)}}) \arrow[r, "\sim"] & (R^if_*\mathcal{O}_X)_y\hat{} \arrow[r] & (R^if_*\mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y \arrow[u] \end{tikzcd}$$

Since s is injective, the map labeled s' is injective as well. $\square$

Remark 3.2. *In item (a), we actually proved the sharper statement*

$$W_0H^i(X,\mathbb{C}) \hookrightarrow Gr_F^0 H^i(X,\mathbb{C}) = im[H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X)]$$

For certain classes of singularities called Du Bois singularities [PS, §7.3.3], *which include rational singularities* [K], $Gr_F^0 H^i(X,\mathbb{C}) = H^i(X,\mathcal{O}_X)$. *But this is not true in general.*

Corollary 3.3. *Suppose that $f : X \to Y$ is a resolution of singularities.*

(1) *If Y has rational singularities then $W_0H^i(f^{-1}(y),\mathbb{C}) = 0$ for $i > 0$.*
(2) *If Y has isolated normal Cohen-Macaulay singularities, $W_0H^i(f^{-1}(y),\mathbb{C}) = 0$ for $0 < i < \dim Y - 1$*

Proof. The first statement is an immediate consequence of the theorem. The second follows from the well known fact given below. We sketch the proof for lack of a suitable reference.

Proposition 3.4. *If $f : X \to Y$ is a resolution of a variety with isolated normal Cohen-Macaulay singularities, then $R^if_*\mathcal{O}_X = 0$ for $0 < i < \dim Y - 1$*

Sketch. We can assume that Y is projective. By the Kawamata-Viehweg vanishing theorem [Ka, V]

$$H^i(X, f^*L^{-1}) = 0, \quad i < \dim Y = n, \tag{4}$$

where L is ample. Replace L by L^N, with $N \gg 0$. Then by Serre vanishing and Serre duality (we use the CM hypothesis here)

$$H^i(Y, L^{-1}) = H^{n-i}(Y, \omega_Y \otimes L) = 0, \quad i < n. \tag{5}$$

The Leray spectral sequence together with (4) and (5) imply

$$H^0(R^if_*\mathcal{O}_X \otimes L^{-1}) = 0, \quad i < n - 1$$

Since the sheaves $R^if_*\mathcal{O}_X$ have zero dimensional support, the proposition follows. □

□

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